

AD-A100 336

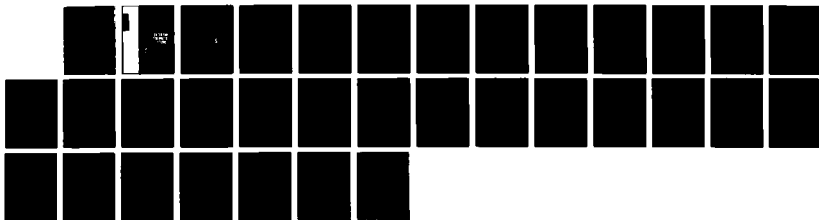
CONE RATIO DATA ENVELOPMENT ANALYSIS AND
MULTI-OBJECTIVE PROGRAMMING(U) TEXAS UNIV AT AUSTIN
CENTER FOR CYBERNETIC STUDIES A CHARNES ET AL. JAN 87
CCS-RR-559 N00014-86-C-0398

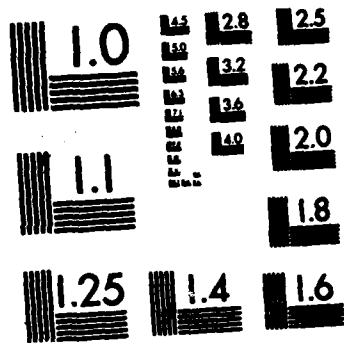
1/1

UNCLASSIFIED

F/G 12/4

NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A180 336

(12)

Research Report CCS 559

CONE RATIO DATA ENVELOPMENT ANALYSIS
AND MULTI-OBJECTIVE PROGRAMMING

by

A. Charnes
W.W. Cooper
Q.L. Wei*
Z.M. Huang

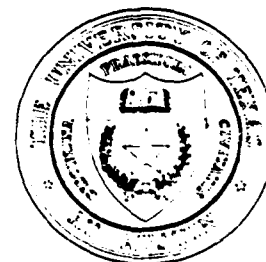
**CENTER FOR
CYBERNETIC
STUDIES**

The University of Texas
Austin, Texas 78712



DISTRIBUTION STATEMENT A

Approved for public release
Distribution Unlimited



87 5 8 057

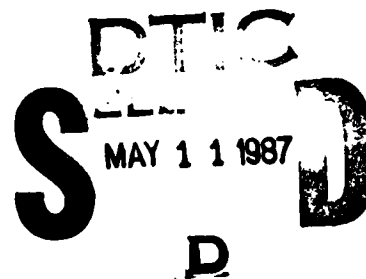
12

Research Report CCS 559
CONE RATIO DATA ENVELOPMENT ANALYSIS
AND MULTI-OBJECTIVE PROGRAMMING

by

A. Charnes
W.W. Cooper
Q.L. Wei*
Z.M. Huang

January 1987



*The People's University of China in Beijing

This research was partly supported by ONR Contracts N00014-86-C-0398 and N00014-82-K-0295, and National Science Foundation Grants SES-8408134 and SES-8520806 with the Center for Cybernetic Studies, The University of Texas at Austin. Reproduction in whole or in part is permitted for any purpose of the United States Government.

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director

College of Business Administration, 5.202
The University of Texas at Austin
Austin, Texas 78712-1177
(512) 471-1821

- A

ABSTRACT

A new "cone-ratio" Data Envelopment Analysis model which substantially generalizes the CCR model and the Charnes-Cooper Thrall approach characterizing its efficiency classes is herein developed and studied. It allows for infinitely many DMU's and arbitrary closed convex cones for the virtual multipliers as well as the cone of positivity of the vectors involved. Generalizations of linear programming and polar cone dualizations are the analytical vehicles employed.

KEY WORDS :

Data Envelopment Analysis
Multi-attribute Optimization
Efficiency Analysis
Cone-Ratio Models
Polar Cones



Accession For	
NTIS	CRA&I <input checked="" type="checkbox"/>
DTIC	TAB <input type="checkbox"/>
Unannounced <input type="checkbox"/>	
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

1. Introduction

We develop the following new "cone-ratio" DEA model which substantially generalizes the CCR model [3] as well as the approach of Charnes, Cooper and Thrall [8] to characterizing its efficiency classes:

$$(C^2WH) \begin{cases} \text{Max } u^T y_{j_0} / v^T x_{j_0} \\ \text{s.t. } v^T \bar{X} - u^T \bar{Y} \in K \\ v \in V, u \in U, (V \neq \emptyset, U \neq \emptyset) \end{cases}$$

where

$V \subset E_+^m$ is a closed convex cone, and $\text{Int } V \neq \emptyset$.

$U \subset E_+^s$ is a closed convex cone, and $\text{Int } U \neq \emptyset$.

$K \subset E^n$ is a closed convex cone, and

$$\delta_j = (0, \dots, 0, \underbrace{1}_j, 0, \dots, 0)^T \in -K^*, \quad j = 1, \dots, n,$$

where $K^* = \{k \mid \hat{k}^T k \leq 0, \forall \hat{k} \in K\}$ is the "polar cone" of the set K .

$\bar{X} = [x_1, \dots, x_n]$ is an $m \times n$ matrix.

$\bar{Y} = [y_1, \dots, y_n]$ is an $s \times n$ matrix.

x_j is the input vector of DMU_j, $x_j \in \text{Int } (-V^*)$.

y_j is the output vector of DMU_j, $y_j \in \text{Int } (-U^*)$.

We shall require the following facts about acute cones. Cone U is said to be an "acute" cone if there exists an open half-space

$$H = \{u \mid a^T u > 0\}$$

such that $\bar{U} \subset H \cup \{0\}$, where \bar{U} is the closure of U . It is easy to prove the following results:

(I) $\text{Int } U^* \neq \emptyset$ if and only if U is an acute cone (See [13]).

(II) When V is an acute cone, $\text{Int } V^* = \{v \mid v^T \hat{v} < 0, \forall \hat{v} \in V, \hat{v} \neq 0\}$ (See [13]).

(III) When V is a closed convex cone and $\text{Int } V \neq \emptyset$, $V^* \cap (-V^*) = \{0\}$.

In Fact, since $(V^*)^* = V$ and $\text{Int } V \neq \emptyset$, V^* is an acute cone. Hence there exists an open half-space $H = \{u: a^T u > 0\}$ such that

$$V^* \subset H \cup \{0\}$$

Namely

$$a^T v^* > 0 \text{ for all nonzero } v^* \in V^*, \quad (1)$$

So

$$a^T \mu^* < 0 \text{ for all nonzero } \mu^* \in -V^*. \quad (2)$$

Combining (1) and (2), we have

$$V^* \cap (-V^*) = \{0\}.$$

We can get $v^T x_{j_0} > 0$ from $x_{j_0} \in \text{Int } (-V^*)$ and $v \in V, v \neq 0$.

Employing the Charnes-Cooper transformation of fractional programming [2],

$$w = tv, \quad \mu = tu, \quad tv^T x_{j_0} = 1$$

we obtain the following pair of dual convex programs as in Ben-Israel, Charnes and Kortanek [12]:

$$\begin{aligned} V_P &= \max \mu^T y_{j_0} \\ (P) \quad &\text{s.t. } w^T \bar{x} - \mu^T \bar{y} \in K, \\ &w^T x_{j_0} = 1, \\ &w \in V, \mu \in U. \end{aligned}$$

and

$$\begin{aligned} V_D &= \min \theta \\ (D) \quad &\text{s.t. } \bar{x}\lambda - \theta x_{j_0} \in V^*, \\ &-\bar{y}\lambda + y_{j_0} \in U^*, \\ &\lambda \in -K^*. \end{aligned}$$

Since $\delta_j \in -K^*$, we can get $K \subset E_+^n$. Therefore

$$V_P = \max \mu^T y_{j_0} \leq w^T x_{j_0} = 1.$$

Definition 1: DMU_{j_0} is said to be "DEA-efficient" if there exists an optimal solution (w^0, μ^0) of program (P) such that

$$\mu^0 T y_{j_0} = 1$$

and

$$w^0 \in \text{Int } V, \mu^0 \in \text{Int } U.$$

Definition 2: DMU_{j₀} is said to be "weak DEA-efficient" if there exists an optimal solution

(w⁰, μ⁰) of program (P) such that

$$\mu^0 T y_{j_0} = 1.$$

The pair of dual programming problems (P) and (D) constitute a model in which convex cones are used to measure the efficiency of DMU's (In the appendix, we present the dual theorem concerning the dual programming problems (P) and (D).) In this paper, we establish the equivalence of DEA efficient solutions and nondominated solutions of multiobjective programming (VP) (see section 2). We also discuss the "projection" of decision making units onto the efficiency surface and the existence of DEA efficiency of DMUs (see section 3).

Let $V = E_+^m$, $U = E_+^s$ and $K = E_+^n$. The pair (P) and (D) is then the CCR model [3]

$$(P1) \begin{cases} V_{P1} = \max \mu^T y_{j_0} \\ \text{s.t. } w^T \bar{X} - \mu^T \bar{Y} \geq 0, \\ w^T x_{j_0} = 1, \\ w, \mu \geq 0. \end{cases}$$

and

$$(D1) \begin{cases} V_{D1} = \min \theta \\ \text{s.t. } \bar{X}\lambda - \theta x_{j_0} \leq 0, \\ -\bar{Y}\lambda + y_{j_0} \leq 0, \\ \lambda \geq 0. \end{cases}$$

If we set $K = E_+^n$ the pair (P) and (D) becomes

$$(P2) \begin{cases} V_{P2} = \max \mu^T y_{j0} \\ \text{s.t. } w^T \bar{X} - \mu^T \bar{Y} \geq 0, \\ \quad \quad \quad w^T x_{j0} = 1 \\ \quad \quad \quad w \in V, \mu \in U. \end{cases}$$

and

$$(D2) \begin{cases} V_{D2} = \min \theta \\ \text{s.t. } \bar{X}\lambda - \theta x_{j0} \in V^*, \\ \quad \quad -\bar{Y}\lambda + y_{j0} \in U^*, \\ \quad \quad \lambda \geq 0. \end{cases}$$

In (P2), the more general conditions $w \in V, \mu \in U$ replace the non-negativity conditions of the CCR model.

If we set $V = E_+^m, U = E_+^s$, we get the pair (P) and (D) as

$$(P3) \begin{cases} V_{P3} = \max \mu^T y_{j0} \\ \text{s.t. } w^T \bar{X} - \mu^T \bar{Y} \in K, \\ \quad \quad \quad w^T x_{j0} = 1, \\ \quad \quad \quad w, \mu \geq 0. \end{cases}$$

and

$$(D3) \begin{cases} V_{D3} = \min \theta \\ \text{s.t. } \bar{X}\lambda - \theta x_{j0} \leq 0, \\ \quad \quad -\bar{Y}\lambda + y_{j0} \leq 0, \\ \quad \quad \lambda \in -K^*. \end{cases}$$

In (D3), we have $\lambda \in -K^*$ which replaces and generalizes the conical hull conditions about the production possibility set in the CCR model [6].

2. DEA Efficiency (or Weak DEA Efficiency) and Nondominated Solutions of Multiobjective Programming Problems

Consider the multiobjective programming problem

$$(VP) \begin{cases} v = \min (f_1(x, y), \dots, f_m(x, y), f_{m+1}(x, y), \dots, f_{m+s}(x, y)) \\ \text{s.t.} \quad (x, y) \in T \end{cases}$$

where

$$T = \{(x, y) : (x, y) \in (\bar{X}\lambda, \bar{Y}\lambda) + (-V^*, U^*), \lambda \in -K^*\}$$

is the production possibility set (It is easy to show that T is a convex cone). Also

$$f_k(x, y) = \begin{cases} x_k, & 1 \leq k \leq m, \\ -y_{k-m}, & m+1 \leq k \leq m+s \end{cases}$$

as in C2GS2, where

$$x = (x_1, \dots, x_k, \dots, x_m)^T,$$

$$y = (y_1, \dots, y_r, \dots, y_s)^T.$$

Since $\delta_j \in -K^*$, we have the input-output vector pairs $(x_j, y_j) \in T$, $j = 1, \dots, n$.

Let

$$f(x, y) = (f_1(x, y), \dots, f_{m+s}(x, y))^T.$$

Definition 3: $(x_{j_0}, y_{j_0}) \in T$ is said to be a nondominated solution of the (VP) associated with $V^* \times U^*$ if there exists no $(x, y) \in T$ such that

$$f(x, y) \in f(x_{j_0}, y_{j_0}) + (V^*, U^*), (x, y) \neq (x_{j_0}, y_{j_0})$$

Namely, there exists no $(x, y) \in T$ such that

$$(x, -y) \in (x_{j_0}, -y_{j_0}) + (V^*, U^*), (x, y) \neq (x_{j_0}, y_{j_0})$$

Definition 4: $(x_{j_0}, y_{j_0}) \in T$ is said to be a nondominated solution of (VP) associated with

$\text{Int } V^* \times \text{Int } U^*$ if there exists no $(x, y) \in T$ such that

$$f(x, y) \in f(x_{j_0}, y_{j_0}) + (\text{Int } V^*, \text{Int } U^*)$$

Namely, there exists no $(x, y) \in T$ such that

$$(x, -y) \in (x_{j_0}, -y_{j_0}) + (\text{Int } V^*, \text{Int } U^*)$$

In this section, we will study the relations between DEA efficiency (or weak DEA efficiency) of DMU's and nondominated solutions of (VP) associated with $V^* \times U^*$ (or $\text{Int } V^* \times \text{Int } U^*$).

Let

$$S = \{(x_j, y_j), j = 1, \dots, n\}$$

$$\tilde{S} = \{(\bar{X}\lambda, \bar{Y}\lambda) : \lambda \in -K^*\}$$

$$T = \{(x, y) : (x, y) \in (\bar{X}\lambda, \bar{Y}\lambda) + (-V^*, U^*), \lambda \in -K^*\}$$

Lemma 1. Let (w^0, μ^0) be an optimal solution of (P), and $\mu^0 T y_{j_0} = 1$. Then for an arbitrary $(x, y) \in T$ we have

$$w^0 T x - \mu^0 T y \geq 0 = w^0 T x_{j_0} - \mu^0 T y_{j_0}.$$

Proof: Since $\mu^0 T y_{j_0} = 1$, we have

$$w^0 T x_{j_0} - \mu^0 T y_{j_0} = 0$$

For an arbitrary $(x, y) \in \tilde{S}$ there exists $\lambda \in -K^*$ such that

$$(x, y) = (\bar{X}\lambda, \bar{Y}\lambda)$$

Since $w^0 T \bar{X} - \mu^0 T \bar{Y} \in K$, then we get

$$w^0 T x - \mu^0 T y = w^0 T \bar{X}\lambda - \mu^0 T \bar{Y}\lambda = (w^0 T \bar{X} - \mu^0 T \bar{Y})\lambda \geq 0.$$

For an arbitrary $(x, y) \in T$, there exists $\lambda \in -K^*$, $v^* \in -V^*$, $u^* \in -U^*$ such that

$$(x, y) = (\bar{X}\lambda + v^*, \bar{Y}\lambda - u^*)$$

So

$$\begin{aligned} w^0 T x - \mu^0 T y &= w^0 T (\bar{X}\lambda + v^*) - \mu^0 T (\bar{Y}\lambda - u^*) \\ &= (w^0 T \bar{X} - \mu^0 T \bar{Y})\lambda + w^0 T v^* + \mu^0 T u^* \geq 0. \end{aligned}$$

Q.E.D.

Theorem 1 Let DMU_{j₀} be DEA efficient. Then (x_{j_0}, y_{j_0}) is a nondominated solution of (VP) associated with $V^* \times U^*$.

Proof: If (x_{j_0}, y_{j_0}) is not a nondominated solution of (VP) associated with $V^* \times U^*$, then there exists $(\bar{x}, \bar{y}) \in T$ such that

$$(\bar{x}, -\bar{y}) \in (x_{j_0}, -y_{j_0}) + (V^*, U^*), (\bar{x}, -\bar{y}) \neq (x_{j_0}, -y_{j_0})$$

that is, there exists $(v^*, u^*) \in (V^*, U^*)$, $(v^*, u^*) \neq 0$ such that

$$(\bar{x}, -\bar{y}) = (x_{j_0}, -y_{j_0}) + (v^*, u^*)$$

Since DMU_{j_0} is DEA efficient, there exists an optimal solution $(w^0, \mu^0) \in \text{Int } V \times \text{Int } U$ such that

$$\mu^0 y_{j_0} = 1.$$

We have

$$\begin{aligned} w^0 \bar{x} - \mu^0 \bar{y} &= (w^0 x_{j_0} - \mu^0 y_{j_0}) + (w^0 v^* + \mu^0 u^*) \\ &< w^0 x_{j_0} - \mu^0 y_{j_0} \end{aligned}$$

as we shall see. For consider $(v^{*T}, u^{*T}) \neq 0$ and without loss of generality, suppose $v^* \neq 0$. Since $w^0 \in \text{Int } V$, $v^* \in V^*$ and V is acute, we have $w^0 v^* < 0$, $\mu^0 u^* \leq 0$, which suffices.

But by Lemma 1, we have

$$w^0 \bar{x} - \mu^0 \bar{y} \geq w^0 x_{j_0} - \mu^0 y_{j_0}$$

a contradiction.

Q.E.D.

Theorem 2. Let (x_{j_0}, y_{j_0}) be a nondominated solution of (VP) associated with $V^* \times U^*$ and let Assumption (A) hold (see Appendix). Then DMU_{j_0} is DEA efficient.

Proof: Since $\tilde{S} \subset T$, the following system (I) is inconsistent:

$$(I) \begin{cases} (\bar{X}\lambda, -\bar{Y}\lambda) \in (x_{j_0}, -y_{j_0}) + (V^*, U^*), (\bar{X}\lambda, -\bar{Y}\lambda) \neq (x_{j_0}, -y_{j_0}) \\ \lambda \in -K^* \end{cases}$$

Now let us consider the pair of dual programming problems

$$(\bar{P}) \begin{cases} V_{\bar{P}} = \min (w^T x_{j_0} - \mu^T y_{j_0}) \\ \text{s.t. } w^T \bar{X} - \mu^T \bar{Y} \in K, \\ w - \tau \in V, \\ \mu - \hat{\tau} \in U. \end{cases}$$

and

$$(\bar{D}) \begin{cases} V_{\bar{D}} = \max (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t. } \bar{X}\lambda - x_{j_0} + s^- = 0, \\ -\bar{Y}\lambda + y_{j_0} + s^+ = 0, \\ \lambda \in -K^*, s^- \in -V^*, s^+ \in -U^*. \end{cases}$$

where $\tau \in \text{Int } V$, $\hat{\tau} \in \text{Int } U$.

First, we want to show $V_{\bar{D}} = 0$. For an arbitrary feasible solution (λ, s^-, s^+) of (D), since $s^- \in -V^*$, $\tau \in \text{Int } V$, $s^+ \in -U^*$, $\hat{\tau} \in \text{Int } U$, then

$$\tau^T s^- \geq 0, \quad \hat{\tau}^T s^+ \geq 0,$$

so $V_{\bar{D}} \geq 0$. If $V_{\bar{D}} > 0$, namely there exists an optimal solution $(\lambda^0, s^{0-}, s^{0+})$ of (D), such that

$$V_{\bar{D}} = \tau^T s^{0-} + \hat{\tau}^T s^{0+} > 0,$$

then we have

$$(x\lambda^0, -\bar{Y}\lambda^0) = (x_{j_0}, -y_{j_0}) + (-s^{0-}, -s^{0+}), \quad (-s^{0-}, -s^{0+}) \in (V^*, U^*), \quad (s^{0-}, s^{0+}) \neq 0$$

This yields a contradiction because (I) is inconsistent.

By the dual theorem (see Appendix, Th. 3), we have $V_{\bar{P}} = 0$.

Secondly, let $(\tilde{w}, \tilde{\mu})$ be an optimal solution of (\bar{P}) , and let

$$w^0 = \tilde{w} / \tilde{w}^T x_{j_0}, \quad \mu^0 = \tilde{\mu} / \tilde{w}^T x_{j_0}$$

Then we have

$$w^0 T x_{j_0} = \mu^0 T y_{j_0} = 1,$$

$$w^0 T \bar{X} - \mu^0 T \bar{Y} \in K$$

$$w^0 \in \tau / \bar{w}^T x_{j_0} + V \subset \text{Int } V \quad (\text{since } \tau \in \text{Int } V)$$

$$\mu^0 \in \hat{\tau} / \bar{w}^T x_{j_0} + U \subset \text{Int } U \quad (\text{since } \hat{\tau} \in \text{Int } U)$$

Namely,

$$\max \mu^1 y_{j_0} - \mu^0 T y_{j_0} = 1,$$

$$w^0 T X - \mu^0 T Y \in K,$$

$$w^0 T x_{j_0} = 1$$

$$w^0 \in \text{Int } V, \mu^0 \in \text{Int } U$$

So DMU_{j_0} is DEA efficient.

Q.E.D.

Theorem 3 Let DMU_{j_0} be weak DEA efficient. Then (x_{j_0}, y_{j_0}) is a nondominated solution of (VP) associated with $\text{Int } V^* \times \text{Int } U^*$.

Its proof is similar to Theorem 1.

Theorem 4 Let (x_{j_0}, y_{j_0}) be a nondominated solution of (VP) associated with $\text{Int } V^* \times \text{Int } U^*$, and Assumption (B) hold (see Appendix). Then DMU_{j_0} is weak DEA efficient.

Proof. Since (x_{j_0}, y_{j_0}) is a nondominated solution of (VP) associated with $\text{Int } V^* \times \text{Int } U^*$, then the following system (II) is inconsistent.

$$(II) \begin{cases} (X\lambda, -\bar{Y}\lambda) \in (x_{j_0}, -y_{j_0}) + (\text{Int } V^*, \text{Int } U^*) \\ \lambda \in -K^* \end{cases}$$

Consider the pair of dual programming problems

$$(\hat{P}) \begin{cases} V_{\hat{P}} = \min (w^T x_{j_0} - \mu^T y_{j_0}) \\ \text{s.t. } w^T \bar{X} - \mu^T \bar{Y} \in K, \\ w - v \in V, \\ \mu - u \in U, \\ \tau^T v + \hat{\tau}^T u = 1, \\ v \in V, u \in U. \end{cases}$$

and

$$(\hat{D}) \begin{cases} V_{\hat{D}} = \max z \\ \text{s.t. } \bar{X}\lambda - x_{j_0} + s^- = 0, \\ -\bar{Y}\lambda + y_{j_0} + s^+ = 0, \\ z\tau - s^- \in V^*, \\ z\hat{\tau} - s^+ \in U^*, \\ \lambda \in -K, s^- \in -V^*, s^+ \in -U^* \end{cases}$$

where $\tau \in \text{Int } V$, $\hat{\tau} \in \text{Int } U$.

Since $\delta_j \in -K^*$, $j = 1, \dots, n$, then

$$(\bar{\lambda}, \bar{s}^-, \bar{s}^+, \bar{z}) = (\delta_{j_0}, 0, 0, 0)$$

is a feasible solution of (\hat{D}) , and

$$V_{\hat{D}} = \max z \geq 0.$$

First, we have to show $V_{\hat{D}} = 0$. If $V_{\hat{D}} > 0$, there exists an optimal solution

$(\lambda^0, s^{0-}, s^{0+}, z^0)$ of (\hat{D}) such that

$$V_{\hat{D}} = \max z = z^0 > 0.$$

Since $V \subset E_+^m$, then

$$\text{Int } V^* = \{w: w^T v < 0, \forall v \in V \text{ and } v \neq 0\}.$$

Because of $z^0 \tau > 0$, we have

$$(-z^0 \tau)^T v < 0, \text{ for all } v \in V \text{ and } v \neq 0.$$

So

$$-z^0\tau \in \text{Int } V^*.$$

Similarly we can show

$$-z^0\hat{\tau} \in \text{Int } U^*.$$

Hence we have

$$-s^{0-} \in V^* - z^0\tau \subset \text{Int } V^*,$$

$$-s^{0+} \in U^* - z^0\hat{\tau} \subset \text{Int } U^*.$$

This yields a contradiction because (II) is inconsistent.

By the dual theorem (see Appendix, Th. 4), we have $V_{\hat{p}} = V_{\hat{d}} = 0$.

Secondly, let $(\bar{w}, \bar{\mu}, \bar{v}, \bar{u})$ be an optimal solution of (\hat{P}) , then we have

$$\bar{w} \in \bar{v} + V \subset V,$$

$$\bar{\mu} \in \bar{u} + U \subset U.$$

Since

$$\bar{w} = \bar{v} + v^{**}, \quad v^{**} \in V$$

$$\bar{\mu} = \bar{u} + u^{**}, \quad u^{**} \in U$$

we have

$$\tau^T \bar{w} + \hat{\tau}^T \bar{\mu} = (\tau^T \bar{v} + \hat{\tau}^T \bar{u}) + (\tau^T v^{**} + \hat{\tau}^T u^{**}) \geq 1.$$

So $(\bar{w}, \bar{\mu}) \neq 0$. Since $V_{\hat{p}} = V_{\hat{d}} = 0$, then we get

$$\bar{w}^T x_{j_0} = \bar{\mu}^T y_{j_0}.$$

Therefore $\bar{w} \neq 0, \bar{\mu} \neq 0$. Let

$$w^0 = \bar{w} / \bar{w}^T x_{j_0}, \quad \mu^0 = \bar{\mu} / \bar{\mu}^T x_{j_0}$$

we have

$$\mu^{0T} y_{j_0} = w^{0T} x_{j_0} = 1,$$

$$w^{0T} \bar{x} - \mu^{0T} \bar{y} \in K,$$

$$w^0 \in \bar{v} / \bar{w}^T x_{j_0} + V \subset V$$

$$\mu^0 \in \bar{u} / \bar{\mu}^T x_{j_0} + U \subset U$$

Namely,

$$\left\{ \begin{array}{l} \max \quad \mu^T y_{j_0} = \mu^{0T} y_{j_0} = 1 \\ \text{s.t.} \quad w^T \bar{X} - \mu^T \bar{Y} \in K, \\ \quad \quad w^T x_{j_0} = 1, \\ \quad \quad w \in V, \quad \mu \in U \end{array} \right.$$

and $w^0 \in V, \mu^0 \in U$. So DMU_{j_0} is weak DEA efficient.

Q.E.D.

3. Efficiency Surface "Projection" and Existence of DEA Efficiency

For an arbitrary $(x_{j_0}, y_{j_0}) \in S = \{(x_j, y_j), j = 1, \dots, n\}$, we consider the following programming problem:

$$(PJ^0) \left\{ \begin{array}{l} \max \quad (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t.} \quad \bar{X}\lambda - x_{j_0} + s^- = 0, \\ \quad \quad -\bar{Y}\lambda + y_{j_0} + s^+ = 0, \\ \quad \quad \lambda \in -K^*, s^- \in -V^*, s^+ \in -U^* \end{array} \right.$$

where $\tau \in \text{Int } V, \hat{\tau} \in \text{Int } U$.

Suppose $(\lambda^0, s^{0-}, s^{0+})$ is an optimal solution of (PJ^0) . Let

$$\hat{x} = \bar{X}\lambda^0 = x_{j_0} - s^{0-},$$

$$\hat{y} = \bar{Y}\lambda^0 = y_{j_0} + s^{0+}.$$

We call (\hat{x}, \hat{y}) the "projection" of DMU_{j_0} onto the efficiency "surface" of the production function (see [4], p 70).

It is obvious that $(\hat{x}, \hat{y}) \in T$. Since $y_{j_0} \in \text{Int } (-U^*), s^{0+} \in -U^*$, we have

$$\hat{y} = y_{j_0} + s^{0+} \in \text{Int } (-U^*).$$

Because $0 \notin \text{Int } (-U^*)$, then we get $\hat{y} \neq 0$. Therefore $(\hat{x}, \hat{y}) \neq 0$.

Theorem 5. The projection (\hat{x}, \hat{y}) of DMU_{j_0} is a nondominated solution of the (VP)

associated with $V^* \times U^*$.

Proof. Suppose (\hat{x}, \hat{y}) is not a nondominated solution of (VP) associated with $V^* \times U^*$.

Then there exists $(\bar{x}, \bar{y}) \in T$ and $(\hat{v}, \hat{u}) \in (V^*, U^*)$ such that

$$(\bar{x}, \bar{y}) = (\hat{x}, \hat{y}) + (\hat{v}, \hat{u}), \quad (\hat{v}, \hat{u}) \neq 0$$

Since $(\bar{x}, \bar{y}) \in T$, there exists $\bar{\lambda} \in -K^*$ and $(\bar{v}, \bar{u}) \in (V^*, U^*)$

such that

$$(\bar{x}, \bar{y}) = (\bar{X}\bar{\lambda}, \bar{Y}\bar{\lambda}) + (-\bar{v}, \bar{u})$$

So we have

$$(\bar{X}\bar{\lambda}, -\bar{Y}\bar{\lambda}) = (\hat{x}, -\hat{y}) + (\hat{v} + \bar{v}, \hat{u} + \bar{u}) \in (\hat{x}, -\hat{y}) + (V^*, U^*) \quad (1)$$

and

$$(\hat{v} + \bar{v}, \hat{u} + \bar{u}) \neq 0 \quad (2)$$

(In fact, if $(\hat{v} + \bar{v}, \hat{u} + \bar{u}) = 0$, we would have $(\bar{v}, \bar{u}) = (\hat{v}, -\hat{u}) \in (V^*, U^*)$)

Since $(\hat{v}, \hat{u}) \neq 0$, without loss of generality, let $\hat{v} \neq 0$. Then we have $\bar{v} = -\hat{v} \in V^*$. This yields a contradiction to $V^* \cap (-V^*) = \{0\}$.

Let

$$v^* = \hat{v} + \bar{v} \in V^*, \quad u^* = \hat{u} + \bar{u} \in U^*.$$

By (1) and (2), we have

$$(\bar{X}\bar{\lambda}, -\bar{Y}\bar{\lambda}) = (\hat{x}, -\hat{y}) + (v^*, u^*), \quad (v^*, u^*) \neq 0$$

so

$$\bar{X}\bar{\lambda} = \hat{x} + v^* = x_{j_0} - s^{0-} + v^*,$$

$$-\bar{Y}\bar{\lambda} = -\hat{y} + u^* = -y_{j_0} - s^{0+} + u^*.$$

Then we get

$$\begin{cases} \bar{X}\bar{\lambda} + (s^{0-} - v^*) = x_{j_0}, \\ -\bar{Y}\bar{\lambda} + (s^{0+} - u^*) = -y_{j_0}, \\ \bar{\lambda} \in -K^*, \quad s^{0-} - v^* \in -V^*, \quad s^{0+} - u^* \in -U^*. \end{cases}$$

Further, since $\tau \in \text{Int } V$, $v^* \in V^*$, $\hat{t} \in \text{Int } U$, $u^* \in U^*$, we have

$$\tau^T v^* \leq 0, \quad \hat{t}^T u^* \leq 0.$$

We know that $(v^*, u^*) \neq 0$, so

$$\tau T v^* + \hat{\tau} T u^* < 0.$$

Thus

$$\begin{aligned} & \tau T(s^{0-} - v^*) + \hat{\tau} T(s^{0+} - u^*) \\ &= (\tau T s^{0-} + \hat{\tau} T s^{0+}) - (\tau T v^* + \hat{\tau} T u^*) \\ &> \tau T s^{0-} + \hat{\tau} T s^{0+}. \end{aligned}$$

This contradicts the fact that $(\lambda^0, s^{0-}, s^{0+})$ is an optimal solution of (PJ^0) . Thus (\hat{x}, \hat{y}) is a nondominated solution of (VP) associated with $V^* \times U^*$.

Q.E.D.

Corollary 1. Let

$$(x_{n+1}, y_{n+1}) = (\hat{x}, \hat{y})$$

where (\hat{x}, \hat{y}) is the projection of DMU_{j_0} . Then DMU_{n+1} is DEA efficient.

Proof: By Theorem 1 and Theorem 2, DEA efficiency and nondominated solution of (VP) are equivalent properties.

Q.E.D.

Theorem 6 Suppose

(I) For arbitrary $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^T \in -K^*$, we have

$$\lambda_j V^* \subset V^*, \quad \lambda_j U^* \subset U^*, \quad j = 1, 2, \dots, n.$$

where

$$\lambda_j V^* = \{\lambda_j v^* : v^* \in V^*\}, \quad \lambda_j U^* = \{\lambda_j u^* : u^* \in U^*\}.$$

(II) For arbitrary $\lambda^l = (\lambda_1^l, \lambda_2^l, \dots, \lambda_n^l)^T \in -K^*$, $l = 0, 1, \dots, n$,

we have

$$(\lambda^1, \lambda^2, \dots, \lambda^n) \lambda^0 = \left(\sum_{k=1}^n \lambda_1^k \lambda_k^0, \sum_{k=1}^n \lambda_2^k \lambda_k^0, \dots, \sum_{k=1}^n \lambda_n^k \lambda_k^0 \right) \in -K^*$$

Then there exists at least one DMU_{j_0} ($1 \leq j_0 \leq n$) which is DEA efficient.

Proof: By Theorem 1 and Theorem 2, it is only necessary to show that there exists some $(x_{j0}, y_{j0}) \in S$ such that it is a nondominated solution of (VP) associated with $V^* \times U^*$.

Suppose for an arbitrary j ($j = 1, \dots, n$), (x_j, y_j) is not a nondominated solution of (VP) associated with $V^* \times U^*$, then there exist $(\bar{x}_j, \bar{y}_j) \in T$ and $\bar{\lambda}^j \in -K^*$ such that

$$(\bar{x}_j, \bar{y}_j) \in (\bar{X} \bar{\lambda}^j, \bar{Y} \bar{\lambda}^j) + (-V^*, U^*) \quad (3)$$

and

$$(x_j, -\bar{y}_j) \in (x_j, -y_j) + (V^*, U^*), \quad (\bar{x}_j, \bar{y}_j) \neq (x_j, y_j), \quad j = 1, 2, \dots, n \quad (4)$$

By (3), there exist $\bar{v}^j \in V^*, \bar{u}^j \in U^*$ such that

$$(\bar{x}_j, \bar{y}_j) = (\bar{X} \bar{\lambda}^j, \bar{Y} \bar{\lambda}^j) + (-\bar{v}^j, \bar{u}^j) \quad (3')$$

By (4), there exist $v \in V^*, u \in U^*$ such that

$$(x_j, y_j) = (\bar{x}_j, \bar{y}_j) + (v^j, -u^j), \quad (v^j, u^j) \neq 0 \quad (4')$$

By Theorem 5, there exists $\lambda^0 \in -K^*, \lambda^0 \neq 0$ such that

$$(\hat{x}, \hat{y}) = (\bar{X} \lambda^0, \bar{Y} \lambda^0) \quad (5)$$

is a nondominated solution of (VP).

Multiplying (4') by λ_j^0 and summing over j , we get

$$\begin{pmatrix} \sum_{j=1}^n \bar{x}_j \lambda_j^0 \\ \sum_{j=1}^n \bar{y}_j \lambda_j^0 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n x_j \lambda_j^0 \\ \sum_{j=1}^n y_j \lambda_j^0 \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n v^j \lambda_j^0 \\ -\sum_{j=1}^n u^j \lambda_j^0 \end{pmatrix}$$

namely,

$$\begin{pmatrix} \sum_{j=1}^n \bar{x}_j \lambda_j^0 \\ -\sum_{j=1}^n \bar{y}_j \lambda_j^0 \end{pmatrix} = \begin{pmatrix} \bar{X} \lambda^0 \\ -\bar{Y} \lambda^0 \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n v^j \lambda_j^0 \\ \sum_{j=1}^n u^j \lambda_j^0 \end{pmatrix} \quad (6)$$

By (6), (5) and assumption (I), we have

$$\begin{pmatrix} \sum_{j=1}^n \bar{x}_j \lambda_j^0 \\ -\sum_{j=1}^n \bar{y}_j \lambda_j^0 \end{pmatrix} = \begin{pmatrix} \hat{x} \\ -\hat{y} \end{pmatrix} + \begin{pmatrix} \sum_{j=1}^n v^j \lambda_j^0 \\ \sum_{j=1}^n u^j \lambda_j^0 \end{pmatrix} \in \begin{pmatrix} \hat{x} \\ -\hat{y} \end{pmatrix} + \begin{pmatrix} V^* \\ U^* \end{pmatrix} \quad (7)$$

By (3'), we have

$$\begin{pmatrix} \sum_{j=1}^n \bar{x}_j \lambda_j^0 \\ \sum_{j=1}^n \bar{y}_j \lambda_j^0 \end{pmatrix} = \begin{pmatrix} \sum_{K=1}^n \left(\sum_{j=1}^n x_j \bar{\lambda}_j^K - \bar{v}^K \right) \lambda_K^0 \\ \sum_{K=1}^n \left(\sum_{j=1}^n y_j \bar{\lambda}_j^K + \bar{u}^K \right) \lambda_K^0 \end{pmatrix} \\ = \begin{pmatrix} \sum_{j=1}^n \left(\sum_{K=1}^n \bar{\lambda}_j^K \lambda_K^0 \right) x_j \\ \sum_{j=1}^n \left(\sum_{K=1}^n \bar{\lambda}_j^K \lambda_K^0 \right) y_j \end{pmatrix} + \begin{pmatrix} -\sum_{K=1}^n \bar{v}^K \lambda_K^0 \\ \sum_{K=1}^n \bar{u}^K \lambda_K^0 \end{pmatrix}$$

By assumption (II), we have

$$\left(\sum_{K=1}^n \bar{\lambda}_1^K \lambda_K^0, \sum_{K=1}^n \bar{\lambda}_2^K \lambda_K^0, \dots, \sum_{K=1}^n \bar{\lambda}_n^K \lambda_K^0 \right)^T \in -K^*$$

By assumption (I), we have

$$\sum_{K=1}^n \bar{v}^K \lambda_K^0 \in V^*, \quad \sum_{K=1}^n \bar{u}^K \lambda_K^0 \in U^*$$

so we get

$$\begin{pmatrix} \sum_{j=1}^n \bar{x}_j \lambda_j^0 \\ \sum_{j=1}^n \bar{y}_j \lambda_j^0 \end{pmatrix} \in T \quad (8)$$

Since $\lambda^0 \neq 0$, then

$$\begin{pmatrix} \sum_{j=1}^n v^j \lambda_j^0 \\ \sum_{j=1}^n u^j \lambda_j^0 \end{pmatrix} \neq 0 \quad (9)$$

In fact, if

$$\begin{pmatrix} \sum_{j=1}^n v^j \lambda_j^0 \\ \sum_{j=1}^n u^j \lambda_j^0 \end{pmatrix} = 0 \quad (10)$$

by $(v^j, u^j) \neq 0$, $j = 1, \dots, n$, and $\lambda^0 \neq 0$, without loss of generality, let $\lambda_{j^*}^0 \neq 0$ and $v^{j^*} \neq 0$. Then by (10), we have

$$\sum_{j \neq j^*} v^j \lambda_j^0 = -v^{j^*} \lambda_{j^*}^0 \neq 0$$

By assumption (1), we get

$$v^{j^*} \lambda_{j^*}^0 \in V^* \cap (-V^*),$$

a contradiction.

By (7), (8) and (9), we get a contradiction to (\hat{x}, \hat{y}) is a nondominated solution of (VF) associated with $V^* \times U^*$.

Q.E.D.

Appendix

Consider the following pair of dual programming problems

$$(P) \begin{cases} \min & c^T x \\ \text{s.t.} & Ax - b \in K \end{cases}$$

and

$$(D) \begin{cases} \max & y^T b \\ \text{s.t.} & y^T A - c^T = 0 \\ & y \in -K^* \end{cases}$$

where A is an $m \times n$ matrix, $b \in E^m$, $c \in E^n$, $K \subset E^m$ is a closed convex cone and $\text{Int } K \neq \emptyset$ (let $K^0 = \text{Int } K$).

Let (see [13], [14] and [15])

$$R = \{x: Ax - b \in K\}$$

$$I(K^0, \bar{z}) = \{z - \alpha \bar{z}: z \in K^0, \alpha \geq 0\}, \quad \bar{z} \in K$$

$$T(R, \bar{x}) = \{z: \exists x^K \in R \text{ and } \alpha_K > 0, \text{ such that } \lim_{K \rightarrow \infty} \alpha_K (x^K - \bar{x}) = z\}$$

$$L(\bar{x}) = \{z: Az \in \overline{I(K^0, Ax - b)}\}$$

$$L^0(\bar{x}) = \text{Int } L(\bar{x})$$

$$D(\bar{x}) = \{-A^T y: y \in -K^*, y^T(A\bar{x} - b) = 0\}$$

where $x \in R$.

It is easy to establish the following lemma:

Lemma 1.

- (i) $I(K^0, \bar{z})$ is an open convex cone.
- (ii) $L(x)$ is a closed convex cone.
- (iii) $D(\bar{x})$ is a convex cone.

Lemma 2. $I^*(K^0, \bar{z}) = \{y: y \in K^*, y^T \bar{z} = 0\}$.

Proof: Let $y \in I^*(K^0, \bar{z})$, then for arbitrary $z \in K^0$ and $\alpha \geq 0$ we have

$$y^T(z - \alpha \bar{z}) \leq 0 \quad (*)$$

Let $\alpha = 0$, we get

$$y^T z \leq 0, \quad \forall z \in K^0.$$

namely, $y \in (K^0)^* = K^*$.

Since $\bar{z} \in K$, we have $y^T \bar{z} \leq 0$. By (*), we get $y^T \bar{z} \geq 0$, so $y^T \bar{z} = 0$.

Therefore

$$I^*(K^0, \bar{z}) \subset \{y: y \in K^*, y^T \bar{z} = 0\}.$$

Let $y \in \{y: y \in K^*, y^T \bar{z} = 0\}$. Then for arbitrary $z \in K^0$, $\alpha \geq 0$, we have

$$\begin{aligned} & y^T(z - \alpha \bar{z}) \\ &= y^T z - \alpha y^T \bar{z} \\ &= y^T z \\ &\leq 0, \end{aligned}$$

so

$$y \in I^*(K^0, \bar{z}).$$

Therefore

$$\{y: y \in K^*, y^T \bar{z} = 0\} \subset I^*(K^0, \bar{z})$$

Q.E.D.

Lemma 3.

$$(I) \quad L(\bar{x}) = D^*(\bar{x}).$$

$$(II) \quad \text{If } D(x) \text{ is closed, then } L^*(\bar{x}) = D(\bar{x}).$$

Proof:

(I) Let $z \in D^*(\bar{x})$, then for an arbitrary

$$y \in I^*(K^0, A\bar{x} - b) = \{y: y \in K^*, y^T(A\bar{x} - b) = 0\},$$

we have $-A^T(-y) \in D(\bar{x})$, hence

$$(Az)^T y = z^T(-A^T(-y)) \leq 0.$$

Therefore

$$Az \in (I^*(K^0, A\bar{x} - b))^* = \overline{I(K^0, A\bar{x} - b)}.$$

Namely,

$$D^*(\bar{x}) \subset L(\bar{x}).$$

Now, let $z \in L(\bar{x})$, i.e.

$$Az \in \overline{I(K^0, A\bar{x} - b)}.$$

Then for arbitrary y satisfying

$$y \in -K^*, \quad y^T(A\bar{x} - b) = 0$$

we have

$$z^T(-A^T y) = (Az)^T(-y) \leq 0$$

(Since $I^*(K^0, A\bar{x} - b) = \{y: y \in K^*, y^T(A\bar{x} - b) = 0\}$, so $-y \in I^*(K^0, A\bar{x} - b)$.) Since $-A^T y \in D(\bar{x})$, we get $z \in D^*(\bar{x})$, namely

$$L(\bar{x}) \subset D^*(\bar{x}).$$

(ii) Since $D(\bar{x})$ is a closed convex cone, from (i) we have

$$L^*(\bar{x}) = D^{**}(\bar{x}) = D(\bar{x}).$$

Q.E.D.

Lemma 4. $T(R, \bar{x}) \subset L(\bar{x})$.

Proof: For an arbitrary $z \in T(R, \bar{x})$, there exist $x^K \in R$ and $\alpha_K > 0$ such that

$$\lim_{K \rightarrow \infty} \alpha_K(x^K - \bar{x}) = z.$$

From $Ax^K - b \in K$ and $K^0 \neq 0$ we know that there exists $\{y^{K,l}\} \subset K^0$ such that

$$\lim_{l \rightarrow \infty} y^{K,l} = Ax^K - b.$$

Because $y^{K,l} \in K^0$ and $\alpha_K > 0$ we have

$$\alpha_K(y^{K,l} - (A\bar{x}^K - b)) \in I(K^0, A\bar{x} - b).$$

Let $l \rightarrow \infty$, we get

$$\alpha_K(Ax^K - b) - \alpha_K(A\bar{x} - b) \in \overline{I(K^0, A\bar{x} - b)}.$$

But

$$A\alpha_K(x^K - \bar{x}) = \alpha_K(Ax^K - b) - \alpha_K(A\bar{x} - b).$$

Thus

$$A\alpha_K(x^K - \bar{x}) \in \overline{I(K^0, A\bar{x} - b)}.$$

Let $K \rightarrow \infty$, we have

$$Az \in \overline{I(K^0, A\bar{x} - b)},$$

namely

$$T(R, \bar{x}) \subset L(\bar{x})$$

Q.E.D

Lemma 5. $L^0(\bar{x}) \subset T(R, \bar{x})$.

Proof: Since $K^0 \neq 0$, it is easy to show that

$$L^0(x) = \{z: Az \in \overline{I(K^0, A\bar{x} - b)}\}.$$

For an arbitrary $z \in L^0(x)$, there exist $u \in K^0$, $\alpha \geq 0$ such that

$$Az = u - \alpha(A\bar{x} - b).$$

Case (i), $\alpha = 0$. For an arbitrary $\beta \geq 0$, we have

$$\begin{aligned} A(\bar{x} + \beta z) - b &= (A\bar{x} - b) + \beta Az \\ &= (A\bar{x} - b) + \beta u \in K \quad (\text{because } \bar{x} \in R \text{ and } u \in K^0). \end{aligned}$$

Take $\{\beta_K\}$ satisfying

$$\beta_1 > \beta_2 > \dots > 0, \quad \lim_{K \rightarrow \infty} \beta_K = 0.$$

Let

$$x^K = \bar{x} + \beta_K z, \quad \alpha_K = 1 - \beta_K,$$

we have $x^K \in R$, $\lim_{K \rightarrow \infty} x^K = \bar{x}$, $\alpha_K > 0$ and

$$z = \alpha_K(x^K - \bar{x}).$$

Therefore

$$z \in T(R, \bar{x}).$$

Case (II), $\alpha > 0$. For an arbitrary $\beta \in [0, 1/\alpha]$ we have

$$\begin{aligned} & A(\bar{x} + \beta z) - b \\ &= A\bar{x} - b + \beta Az \\ &= (A\bar{x} - b) + \beta(u - \alpha(A\bar{x} - b)) \\ &= (1 - \alpha\beta)(A\bar{x} - b) + \beta u \in K \quad (\text{because } \bar{x} \in R, u \in K^0). \end{aligned}$$

Take $\{\beta_K\}$ satisfying $1/\alpha \geq \beta_1 > \beta_2 > \dots > 0$, $\lim_{K \rightarrow \infty} \beta_K = 0$.

Let

$$x^K = \bar{x} + \beta_K z, \quad \alpha_K = 1/\beta_K$$

We have $x^K \in R$, $\alpha_K > 0$, $\lim_{K \rightarrow \infty} x^K = \bar{x}$ and $z = \alpha_K(x^K - \bar{x})$

Therefore

$$z \in T(R, \bar{x}).$$

Q.E.D.

Theorem 1. (Weak Duality Theorem) Let x be a feasible solution of (P), y be a feasible solution of (D). Then

$$c^T x \geq y^T b.$$

Proof. Since $Ax - b \in K$, there exists $u \in K$ such that $Ax = b + u$, hence

$$\begin{aligned} c^T x &= y^T Ax \\ &= y^T (b + u) \\ &\geq y^T b \end{aligned}$$

Q.E.D.

Lemma 6. Let $\bar{x} \in R$ be an optimal solution of (P). Then

$$-c \in T^*(R, \bar{x}).$$

Proof. It is only necessary to show

$$c^T z \geq 0, \quad \text{for } \forall z \in T(R, \bar{x}).$$

Now for an arbitrary $z \in T(R, \bar{x})$, there exist $\{x^K\} \subset R$, $\alpha_K > 0$ and $\lim_{K \rightarrow \infty} x^K = \bar{x}$

such that

$$\lim_{K \rightarrow \infty} \alpha_K(x^K - \bar{x}) = z.$$

Since \bar{x} is an optimal solution of (P), we have

$$c^T \alpha_K(x^K - \bar{x}) = \alpha_K(c^T x^K - c^T \bar{x}) \geq 0.$$

Let $k \rightarrow \infty$, we have

$$c^T z \geq 0.$$

Q.E.D.

Lemma 7. Let $\bar{x} \in R$ be an optimal solution of (P) and let $D(\bar{x})$ be a closed set. Then

$$-c \in D(\bar{x}).$$

Proof. From Lemma 3, Lemma 4 and Lemma 5 we get

$$L^0(\bar{x}) \subset T(R, \bar{x}) \subset L(\bar{x}) = D^*(x),$$

hence

$$L^*(\bar{x}) = (L^0(\bar{x}))^* \supset T^*(R, \bar{x}) \supset L^*(\bar{x}) = D^{**}(\bar{x}) = D(x).$$

Thus

$$L^*(\bar{x}) = T^*(R, \bar{x}) = D(\bar{x}).$$

From Lemma 6, we get

$$-c \in D(\bar{x}).$$

Q.E.D.

Theorem 2. (Dual Theorem) Let $\bar{x} \in R$ be an optimal solution of (P) and let $D(x)$ be a closed set. Then (D) has an optimal solution \bar{y} , and $c^T \bar{x} = \bar{y}^T b$.

Proof. By Lemma 6, we have

$$-c \in D(\bar{x}).$$

Namely, there exists $\bar{y} \in E^m$ such that

$$\bar{y} \in -K^*,$$

$$\bar{y}^T (A\bar{x} - b) = 0,$$

$$-c = -A^T \bar{y}.$$

Therefore

$$\begin{cases} A\bar{x} - b \in K, \\ \bar{y}^T A - c^T = 0, \quad \bar{y} \in -K^* \end{cases}$$

and

$$c^T \bar{x} = \bar{y}^T A \bar{x} = \bar{y}^T b.$$

By Theorem 1, \bar{y} is an optimal solution of (D), and

$$c^T \bar{x} = \bar{y}^T b.$$

Q.E.D.

Note: Take $K = E_+^m$ (namely, (P) and (D) are linear programming problems). Let

$$I = \{i: a_i \bar{x} = b_i, \quad 1 \leq i \leq m\},$$

then

$$D(x) = \left\{ \sum_{i \in I} y_i a_i^T: y_i \geq 0, \quad i \in I \right\},$$

where

$$A = (a_1, a_2, \dots, a_m), \quad b = (b_1, b_2, \dots, b_m)$$

It is easy to show that $D(\bar{x})$ is a closed set.

Let us consider the following pair of dual programs:

$$(P) \begin{cases} \min & (w^T x_{j_0} - \mu^T y_{j_0}) \\ \text{s.t.} & w^T \bar{x} - \mu^T \bar{y} \in K \\ & w - \tau \in V \\ & \mu - \hat{\tau} \in U \end{cases}$$

and

$$(D) \begin{cases} \max & (\tau^T s^- + \hat{\tau}^T s^+) \\ \text{s.t.} & \bar{x} \lambda - x_{j_0} + s^- = 0 \\ & -\bar{y} + y_{j_0} + s^+ = 0 \\ & \lambda \in -K^*, \quad s^- \in -V^*, \quad s^+ \in -U^*. \end{cases}$$

Let $(\lambda^0, s^{0-}, s^{0+})$ be a feasible solution of (\bar{D}) and

$$\bar{D}(\lambda^0, s^{0-}, s^{0+}) = \left\{ \begin{pmatrix} \bar{\lambda}^T w - \bar{\gamma}^T \mu + y_1 \\ w + y_2 \\ \mu + y_3 \end{pmatrix} : \begin{array}{l} y_1 \in K, y_2 \in V, y_3 \in U \\ y_1^T \lambda^0 = y_2^T s^{0-} = y_3^T s^{0+} = 0 \end{array} \right\}$$

Assumption (A): $\bar{D}(\lambda^0, s^{0-}, s^{0+})$ is a closed set.

Theorem 3 Let $(\lambda^0, s^{0-}, s^{0+})$ be an optimal solution of (\bar{D}) and let Assumption (A) hold

Then (\bar{P}) has an optimal solution (w^0, μ^0) , and

$$w^{0T} x_{j_0} - \mu^{0T} y_{j_0} = \tau^T s^{0-} + \hat{\tau}^T s^{0+}.$$

Proof Since the dual of (\bar{D}) is (\bar{P}) , and Assumption (A) holds. By Theorem 2, we can get the results.

Q.E.D

Now let us consider the following pair of dual programs.

$$(\hat{P}) \left\{ \begin{array}{ll} \min & (w^T x_{j_0} - \mu^T y_{j_0}) \\ \text{s.t.} & w^T \bar{X} - \mu^T \bar{Y} \in K \\ & w - v \in V \\ & \mu - u \in U \\ & \tau^T v + \hat{\tau}^T u = 1 \\ & v \in V, u \in U \end{array} \right.$$

and

$$(\hat{D}) \left\{ \begin{array}{ll} \max & z \\ \text{s.t.} & \bar{X}\lambda - x_{j_0} + s^- = 0 \\ & -\bar{Y}\lambda + y_{j_0} + s^+ = 0 \\ & z\tau - s^- \in V^* \\ & z\hat{\tau} - s^+ \in U^* \\ & \lambda \in -K^*, s^- \in -V^*, s^+ \in -U^* \end{array} \right.$$

Let $(\lambda^0, s^{0-}, s^{0+}, z^0)$ be a feasible solution of (\hat{D}) and

$$\hat{D}(\lambda^0, s^{0-}, s^{0+}, z^0) = \left\{ \begin{array}{l} \bar{x}^T w - \bar{y}^T \mu + y_1 \\ w - v + y_2 \\ \mu - u + y_3 \\ \tau^T v + \hat{\tau}^T u \end{array} \right\} \cdot \left. \begin{array}{l} v \in -V, u \in -U \\ y_1 \in K, y_2 \in V, y_3 \in U \\ v^T(z^0 \tau - s^{0-}) = 0 \\ u^T(s^{0+} \hat{\tau} - s^{0+}) = 0 \\ y_1^T \lambda^0 = y_2^T s^{0-} = y_3^T s^{0+} = 0 \end{array} \right\}$$

Assumption (B): $\hat{D}(\lambda^0, s^{0-}, s^{0+}, z^0)$ is a closed set.

Theorem 4. Let $(\lambda^0, s^{0-}, s^{0+}, z^0)$ be an optimal solution of (\hat{D}) , and let Assumption (B)

hold. Then (\hat{P}) has an optimal solution (w^0, μ^0, v^0, u^0) and

$$w^{0T} x_{j_0} - \mu^{0T} y_{j_0} = z^0.$$

Proof It is similar to the proof of Theorem 3.

Q.E.D.

References

- [1] A. Charnes and W.W. Cooper, Management Models and Industrial Applications of Linear Programming, Wiley, New York, 1961.
- [2] A. Charnes and W.W. Cooper, Programming with linear fractional functionals. Naval Research Logistics Quarterly, 9(1962) 181-185.
- [3] A. Charnes, W.W. Cooper and E. Rhodes, Measuring the efficiency of decision making units, European Journal of Operational Research, 2(1978), 429-444.
- [4] A. Charnes and W.W. Cooper, Preface to topics in Data Envelopment Analysis, Annals of Operations Research, 2(1985), 59-94.
- [5] A. Charnes, W.W. Cooper, A.Y. Lewin, R.C. Morey and J. Rousseau, Sensitivity and stability analysis in DEA, Annals of Operations Research, 2(1985), 139-156.
- [6] A. Charnes, W.W. Cooper, B. Golany, L. Seiford and J. Stutz, Foundations of Data Envelopment Analysis for Pareto-Koopmans efficient empirical production functions, Journal of Econometrics, 30(1985).
- [7] R.D. Banker, A. Charnes and W. W. Cooper, Some models for estimating technical and scale inefficiencies in Data Envelopment Analysis, Management Science, 30(1984)9.
- [8] A. Charnes, W.W. Cooper and R. M. Thrall, Characterization of classes in CCT efficiency analysis, Research Report CCS 525, Center for Cybernetic Studies, The University of Texas at Austin, 1985.
- [9] A. Charnes, W.W. Cooper and Q.L. Wei, A semi-Infinite multicriteria programming approach to Data Envelopment Analysis with infinitely many decision-

making units, Research Report 551, Center for Cybernetic Studies, The University of Texas at Austin, 1986.

- [10] C.F. Ku and Q.L. Wei, Problems on MCDM, Applied Mathematics and Computational Mathematics, 1(1980), 28-48, China.
- [11] Q.L. Wei, R.S. Wang, B.Xu, J.Y. Wang and W.L. Bai, Mathematical Programming and Optimum Designs, National Defence and Industry Press, 1984. China.
- [12] A. Ben-Israel, A. Charnes and K.O. Kortanek, Duality and asymptotic solvability over cones, Bulletin of the American Mathematical Society, 75 (1969), 318-324.
- [13] P.L. Yu, Cone convexity, cone extreme point, and nondominated solutions in decision problems with multiobjectives. Journal of Optimization Theory and Applications, 14(1974)3.
- [14] Z.M. Huang, The second order conditions of nondominated solutions for generalized multiobjective mathematical programming, Journal of Systems Science and Mathematical Sciences, 5(1985)3.
- [15] M. Avriel, Nonlinear Programming: Analysis and Methods, Prentice-Hall, Inc., 1976.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER CCS 559	2. GOVT ACCESSION NO. AD-A180336	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CONE RATIO DATA ENVELOPMENT ANALYSIS AND MULTI-OBJECTIVE PROGRAMMING		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER CCS 559
7. AUTHOR(s) A. Charnes, W.W. Cooper, Q.L. Wei, Z.M. Huang		8. CONTRACT OR GRANT NUMBER(s) N00014-86-C-0398 N00014-82-K-0295
9. PERFORMING ORGANIZATION NAME AND ADDRESS Office of Naval Research (Code 434) Washington, DC		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS		12. REPORT DATE January 1987
		13. NUMBER OF PAGES 28
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) This document has been approved for public release and sale; its distribution is unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Data Envelopment Analysis, Multi-attribute Optimization, Efficiency Analysis, Cone-Ratio Models, Polar Cones		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A new "cone-ratio" Data Envelopment Analysis model which substantially generalizes the CCR model and the Charnes-Cooper Thrall approach characterizing its efficiency classes is herein developed and studied. It allows for infinitely many DMU's and arbitrary closed convex cones for the virtual multipliers as well as the cone of positivity of the vectors involved. Generalizations of linear programming and polar cone dualizations are the analytical vehicles employed.		

DD FORM 1 JAN 73 1473

EDITION OF 1 NOV 68 IS OBSOLETE
S/N 0102-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

7-87

DTIC